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Inverse Trigonometric Functions

TOPIC 1 **Trigometric Functions & Their Inverses, Domain & Range of Inverse Trigonometric Functions, Principal Value of Inverse Trigonometric Functions, Intervals for Inverse Trigonometric Functions**

1. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$, $\beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to : **[April 8, 2019 (I)]**

(a) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

(c) $\tan^{-1}\left(\frac{9}{14}\right)$ (d) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

2. A value of x satisfying the equation $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1}x]$, is : **[Online April 9, 2017]**

(a) $-\frac{1}{2}$ (b) -1 (c) 0 (d) $\frac{1}{2}$

3. The principal value of $\tan^{-1}\left(\cot\frac{43\pi}{4}\right)$ is: **[Online April 19, 2014]**

(a) $-\frac{3\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $-\frac{\pi}{4}$ (d) $\frac{\pi}{4}$

4. The number of solutions of the equation, $\sin^{-1}x = 2\tan^{-1}x$ (in principal values) is : **[Online April 22, 2013]**

(a) 1 (b) 4 (c) 2 (d) 3

5. A value of $\tan^{-1}\left(\sin\left(\cos^{-1}\left(\sqrt{\frac{2}{3}}\right)\right)\right)$ is **[Online May 19, 2012]**

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{6}$

6. The largest interval lying in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for which the function, $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$, is defined, is **[2007]**

(a) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left[0, \frac{\pi}{2}\right)$

(c) $[0, \pi]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

7. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is

(a) $[1, 2]$ (b) $[2, 3)$ **[2004]**

(c) $[1, 2]$ (d) $[2, 3]$

8. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for **[2003]**

(a) $|a| \le \frac{1}{\sqrt{2}}$ (b) $\frac{1}{2} < |a| < \frac{1}{\sqrt{2}}$

(c) all real values of a (d) $|a| < \frac{1}{2}$

9. $\cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha}) = x$, then $\sin x =$

(a) $\tan^2\left(\frac{\alpha}{2}\right)$ (b) $\cot^2\left(\frac{\alpha}{2}\right)$ **[2002]**

(c) $\tan\ \alpha$ (d) $\cot\left(\frac{\alpha}{2}\right)$

10. The domain of $\sin^{-1}[\log_3(x/3)]$ is **[2002]**

(a) $[1, 9]$ (b) $[-1, 9]$

(c) $[-9, 1]$ (d) $[-9, -1]$

TOPIC 2 Properties of Inverse Trigonometric Functions, Infinite Series of Inverse Trigonometric Functions

11. $2\pi-\left(\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}\right)$ is equal to : **[Sep. 03, 2020 (I)]**

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) $\frac{7\pi}{4}$

12. If S is the sum of the first 10 terms of the series $\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)+\tan^{-1}\left(\frac{1}{21}\right)+\ldots,$ then tan (S) is equal to: **[Sep. 05, 2020 (I)]**

(a) $\frac{5}{6}$ (b) $\frac{5}{11}$

(c) $-\frac{6}{5}$ (d) $\frac{10}{11}$

13. The value of $\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)$ is equal to : **[April 12, 2019 (I)]**

(a) $\pi-\sin^{-1}\left(\frac{63}{65}\right)$ (b) $\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$

(c) $\frac{\pi}{2}-\cos^{-1}\left(\frac{9}{65}\right)$ (d) $\pi-\cos^{-1}\left(\frac{33}{65}\right)$

14. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, where $-1\le x\le 1, -2\le y\le 2$, $x\le\frac{y}{2}$, then for all x, y, $4x^2-4xy\cos\alpha+y^2$ is equal to: **[April 10, 2019 (II)]**

(a) $4\sin^2\alpha$ (b) $2\sin^2\alpha$

(c) $4\sin^2\alpha-2x^2y^2$ (d) $4\cos^2\alpha+2x^2y^2$

15. Considering only the principal values of inverse functions, the set $A=\left\{x\ge 0:\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}\right\}$ **[Jan. 12, 2019 (I)]**

(a) contains two elements

(b) contains more than two elements

(c) is a singleton

(d) is an empty set

16. All x satisfying the inequality $(\cot^{-1}x)^2-7(\cot^{-1}x)+10>0$, lie in the interval : **[Jan. 11, 2019 (II)]**

(a) $(-\infty,\cot 5)\cup(\cot 4,\cot 2)$

(b) $(\cot 2,\infty)$

(c) $(-\infty,\cot 5)\cup(\cot 2,\infty)$

(d) (cot 5, cot 4)

17. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is: **[Jan. 10, 2019 (II)]**

(a) $\frac{21}{19}$ (b) $\frac{19}{21}$

(c) $\frac{22}{23}$ (d) $\frac{23}{22}$

18. If $x=\sin^{-1}(\sin 10)$ and $y=\cos^{-1}(\cos 10)$, then $y-x$ is equal to: **[Jan. 09, 2019 (II)]**

(a) 0 (b) 10 (c) 7π (d) π

19. If $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}\left(x>\frac{3}{4}\right)$, then x is equal to: **[Jan. 09, 2019 (I)]**

(a) $\frac{\sqrt{145}}{12}$ (b) $\frac{\sqrt{145}}{10}$ (c) $\frac{\sqrt{146}}{12}$ (d) $\frac{\sqrt{145}}{11}$

20. The value of $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$, $|x|<\frac{1}{2}, x\neq 0$, is equal to **[Online April 8, 2017]**

(a) $\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$ (b) $\frac{\pi}{4}+\cos^{-1}x^2$

(c) $\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x^2$ (d) $\frac{\pi}{4}-\cos^{-1}x^2$

21. Let $\tan^{-1}y=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where or $|x|<\frac{1}{\sqrt{3}}$. Then a value of y is : **[2015]**

(a) $\frac{3x-x^3}{1+3x^2}$ (b) $\frac{3x+x^3}{1+3x^2}$

(c) $\frac{3x-x^3}{1-3x^2}$ (d) $\frac{3x+x^3}{1-3x^2}$

22. If $f(x)=2\tan^{-1}x+\sin^{-1}\left(\frac{2x}{1+x^2}\right), x>1$ then $f(5)$ is equal to : **[Online April 10, 2015]**

(a) $\tan^{-1}\left(\frac{65}{156}\right)$ (b) $\frac{\pi}{2}$

(c) π (d) $4\tan^{-1}(5)$

23. **Statement I:** The equation $(\sin^{-1}x)^3+(\cos^{-1}x)^3-a\pi^3=0$ has a solution for all $a\geq\frac{1}{32}$.

Statement II: For any $x\in R$, $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$ and $0\leq\left(\sin^{-1}x-\frac{\pi}{4}\right)^2\leq\frac{9\pi^2}{16}$ **[Online April 12, 2014]**

(a) Both statements I and II are true.
(b) Both statements I and II are false.
(c) Statement I is true and statement II is false.
(d) Statement I is false and statement II is true.

24. If x, y, z are in A.P. and $\tan^{-1}x$, $\tan^{-1}y$ and $\tan^{-1}z$ are also in A.P., then **[2013]**

(a) $x=y=z$ (b) $2x=3y=6z$
(c) $6x=3y=2z$ (d) $6x=4y=3z$

25. Let $x\in(0,1)$. The set of all x such that $\sin^{-1}x>\cos^{-1}x$, is the interval: **[Online April 25, 2013]**

(a) $\left(\frac{1}{2},\frac{1}{\sqrt{2}}\right)$ (b) $\left(\frac{1}{\sqrt{2}},1\right)$
(c) $(0,1)$ (d) $\left(0,\frac{\sqrt{3}}{2}\right)$

26. $S=\tan^{-1}\left(\frac{1}{n^2+n+1}\right)+\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)+\ldots+\tan^{-1}\left(\frac{1}{1+(n+19)(n+20)}\right)$, then $\tan S$ is equal to : **[Online April 23, 2013]**

(a) $\frac{20}{401+20n}$ (b) $\frac{n}{n^2+20n+1}$
(c) $\frac{20}{n^2+20n+1}$ (d) $\frac{n}{401+20n}$

27. A value of x for which $\sin(\cot^{-1}(1+x))=\cos(\tan^{-1}x)$, is : **[Online April 9, 2013]**

(a) $-\frac{1}{2}$ (b) 1 (c) 0 (d) $\frac{1}{2}$

28. If $\sin^{-1}\left(\frac{x}{5}\right)+\operatorname{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$, then the values of x is **[2007]**

(a) 4 (b) 5
(c) 1 (d) 3

29. If $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$, then $4x^2-4xy\cos\alpha+y^2$ is equal to **[2005]**

(a) $2\sin 2\alpha$ (b) 4
(c) $4\sin^2\alpha$ (d) $-4\sin^2\alpha$

Hints & Solutions

1. **(d)** $\because \cos\alpha = \frac{3}{5}$, then $\sin\alpha = \frac{4}{5}$

$\Rightarrow \tan\alpha = \frac{4}{3}$

and $\tan\beta = \frac{1}{3}$

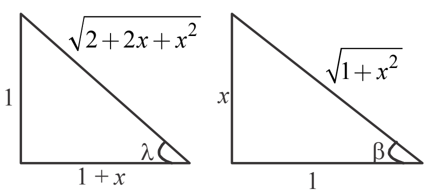
$\because \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha.\tan\beta}$

$$= \frac{\frac{4}{3}-\frac{1}{3}}{1+\frac{4}{9}} = \frac{1}{\frac{13}{9}} = \frac{9}{13}$$

$\therefore \alpha-\beta = \tan^{-1}\left(\frac{9}{13}\right) = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

$= \cos^{-1}\left(\frac{13}{5\sqrt{10}}\right)$

2. **(a)** $\sin\left[\cot^{-1}(1+x)\right] = \cos\left(\tan^{-1}x\right)$



Let; $\cot\lambda = 1+x$
$\tan\beta = x$
$\Rightarrow \sin\lambda = \cos\beta$

$\Rightarrow \frac{1}{\sqrt{x^2+2x+2}} = \frac{1}{1\sqrt{1+x^2}}$

$\Rightarrow x^2+2x+2 = x^2+1$

$\Rightarrow x = -1/2$

3. **(c)** Consider

$\tan^{-1}\left[\cot\frac{43\pi}{4}\right] = \tan^{-1}\left[\cot\left(10\pi+\frac{3\pi}{4}\right)\right]$

$= \tan^{-1}\left[\cot\frac{3\pi}{4}\right]$ $[\because \cot(2n\pi+\theta) = \cot\theta]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{2}-\frac{3\pi}{4}\right)\right]$

$= \frac{\pi}{2}-\frac{3\pi}{4} = \frac{2\pi-3\pi}{4} = \frac{-\pi}{4}$

4. **(a)** Given equation is
$\sin^{-1}x = 2\tan^{-1}x$
Now, this equation has only one solution.

$\therefore$ LHS $= \sin^{-1}1 = \frac{\pi}{2}$

and RHS $= 2\tan^{-1}1 = 2\times\frac{\pi}{4} = \frac{\pi}{2}$

Also, $x = 1$ gives angle value as $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

$\frac{5\pi}{4}$ is outside the principal value.

5. **(d)** Consider $\tan^{-1}\left[\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right]$

Let $\cos^{-1}\sqrt{\frac{2}{3}} = \theta \Rightarrow \cos\theta = \sqrt{\frac{2}{3}}$

$\Rightarrow \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{2}{3}} = \sqrt{\frac{1}{3}}$

$\therefore \tan^{-1}\left[\sin\left(\cos^{-1}\sqrt{\frac{2}{3}}\right)\right] = \tan^{-1}[\sin\theta]$

$= \tan^{-1}\left[\sqrt{\frac{1}{3}}\right] = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

6. **(b)** Given that

$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2}-1\right) + \log(\cos x)$

$f(x)$ is defined if $-1 \le \left(\frac{x}{2}-1\right) \le 1$ and $\cos x > 0$

$\Rightarrow 0 \le \frac{x}{2} \le 2$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\Rightarrow 0 \le x \le 4$ and $-\frac{\pi}{2} < x < \frac{\pi}{2}$

$\therefore x \in \left[0, \frac{\pi}{2}\right)$

7. (b) $f(x)=\dfrac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is defined

When $-1\le x-3\le 1 \Rightarrow 2\le x\le 4$ ....(i)

and $9-x^2>0 \Rightarrow -3<x<3$ ....(ii)

from (i) and (ii),

we get $2\le x<3$ $\therefore$ Domain $=[2,3)$

8. (a) Given that $\sin^{-1}x=2\sin^{-1}a$

We know that $-\dfrac{\pi}{2}\le \sin^{-1}x\le\dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{2}\le 2\sin^{-1}a\le\dfrac{\pi}{2}$

$\Rightarrow -\dfrac{\pi}{4}\le \sin^{-1}a\le\dfrac{\pi}{4}\Rightarrow \dfrac{-1}{\sqrt2}\le a\le\dfrac{1}{\sqrt2}$

$\therefore\ |a|\le\dfrac{1}{\sqrt2}$

9. (a) Given that, $\cot^{-1}(\sqrt{\cos\alpha})-\tan^{-1}(\sqrt{\cos\alpha})=x$

$\tan^{-1}\left(\dfrac{1}{\sqrt{\cos\alpha}}\right)-\tan^{-1}(\sqrt{\cos\alpha})=x$

$\Rightarrow\ \tan^{-1}\dfrac{\dfrac{1}{\sqrt{\cos\alpha}}-\sqrt{\cos\alpha}}{1+\dfrac{1}{\sqrt{\cos\alpha}}.\sqrt{\cos\alpha}}=x$

$\Rightarrow\ \tan^{-1}\dfrac{1-\cos\alpha}{2\sqrt{\cos\alpha}}=x$

$\Rightarrow\ \tan x=\dfrac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

$\Rightarrow\ \cot x=\dfrac{2\sqrt{\cos\alpha}}{1-\cos\alpha}=\dfrac{B}{P}$

$P=(1-\cos\alpha)$ and $B=2\sqrt{\cos\alpha}$

$H=\sqrt{P^2+B^2}=1+\cos\alpha$

$\Rightarrow \sin x=\dfrac{1-\cos\alpha}{1+\cos\alpha}=\dfrac{1-(1-2\sin^2\alpha/2)}{1+2\cos^2\alpha/2-1}$

or $\sin x=\tan^2\dfrac{\alpha}{2}$

10. (a) $f(x)=\sin^{-1}\left(\log_3\left(\dfrac{x}{3}\right)\right)$

We know that domain of $\sin^{-1}x$ is $x\in[-1,1]$

$\therefore\ -1\le\log_3\left(\dfrac{x}{3}\right)\le 1\Rightarrow 3^{-1}\le\dfrac{x}{3}\le 3^1$

$\Rightarrow 1\le x\le 9$ or $x\in[1,9]$

11. (c) $2\pi-\left(\sin^{-1}\dfrac{4}{5}+\sin^{-1}\dfrac{5}{13}+\sin^{-1}\dfrac{16}{65}\right)$

$=2\pi-\left(\tan^{-1}\dfrac{4}{3}+\tan^{-1}\dfrac{5}{12}+\tan^{-1}\dfrac{16}{63}\right)$

$\left[\because \sin^{-1}\dfrac{4}{5}=\tan^{-1}\dfrac{4}{3}\right]$

$=2\pi-\left\{\tan^{-1}\left(\dfrac{\dfrac{4}{3}+\dfrac{5}{12}}{1-\dfrac{4}{3}\cdot\dfrac{5}{12}}\right)+\tan^{-1}\dfrac{16}{63}\right\}$

$=2\pi-\left(\tan^{-1}\dfrac{63}{16}+\tan^{-1}\dfrac{16}{63}\right)$

$=2\pi-\left(\tan^{-1}\dfrac{63}{16}+\cot^{-1}\dfrac{63}{16}\right)$

$=2\pi-\dfrac{\pi}{2}=\dfrac{3\pi}{2}.$

12. (a) $S=\tan^{-1}\left(\dfrac{1}{3}\right)+\tan^{-1}\dfrac{1}{7}+\tan^{-1}\dfrac{1}{13}+....$ upto 10 terms

$=\tan^{-1}\left(\dfrac{2-1}{1+2\cdot1}\right)+\tan^{-1}\left(\dfrac{3-2}{1+3\cdot2}\right)$

$+\tan^{-1}\left(\dfrac{4-3}{1+3\cdot4}\right)+......+\tan^{-1}\left(\dfrac{11-10}{1+11\cdot10}\right)$

$=(\tan^{-1}2-\tan^{-1}1)+(\tan^{-1}3-\tan^{-1}2)+$

$(\tan^{-1}4-\tan^{-1}3)+....+(\tan^{-1}11-\tan^{-1}10)$

$=\tan^{-1}11-\tan^{-1}1=\tan^{-1}\left(\dfrac{11-1}{1+11\cdot1}\right)=\tan^{-1}\left(\dfrac{5}{6}\right)$

$\therefore\ \tan(S)=\dfrac{5}{6}$

13. (b) $-\sin^{-1}\left(\dfrac{3}{5}\right)+\sin^{-1}\left(\dfrac{12}{13}\right)=-\sin^{-1}\left(\dfrac{3}{5}\times\dfrac{5}{13}-\dfrac{12}{13}\times\dfrac{4}{5}\right)$

(∵ xy e" 0 and x^2+y^2 d" 1)

$\left[\because \sin^{-1}x-\sin^{-1}y=\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\}\right]$

$=\sin^{-1}\left(\dfrac{-33}{65}\right)=\sin^{-1}\left(\dfrac{33}{65}\right)$

$=\cos^{-1}\left(\dfrac{56}{65}\right)=\dfrac{\pi}{2}-\sin^{-1}\left(\dfrac{56}{65}\right)$

14. **(a)** Given, $\cos^{-1}x - \cos^{-1}\dfrac{y}{2} = \alpha$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{2} + \sqrt{1-x^2}.\sqrt{1-\frac{y^2}{4}}\right) = \alpha$$

$$\Rightarrow \frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2} = \cos\theta$$

$$\Rightarrow xy + \sqrt{1-x^2}\sqrt{4-y^2} = 2\cos\alpha$$

$\Rightarrow (xy - 2\cos\alpha)^2 = (1-x^2)(4-y^2)$

$\Rightarrow x^2y^2 + 4\cos^2\alpha - 4xy\cos\alpha = 4 - y^2 - 4x^2 + x^2y^2$

$\Rightarrow 4x^2 - 4xy\cos\alpha + y^2 = 4\sin^2\alpha$

15. **(c)** Consider, $\tan^{-1}(2x) + \tan^{-1}(3x) = \dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{1-6x^2} = 1 \Rightarrow 5x = 1 - 6x^2$$

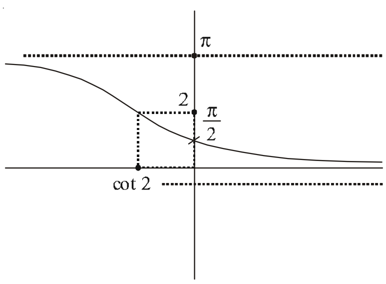
$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow (6x-1)(x+1) = 0$

$\Rightarrow x = \dfrac{1}{6}\ (\text{as } x \geq 0)$

Therefore, A is a singleton set.

16. **(b)**



$(\cot^{-1}x)^2 - 7(\cot^{-1}x) + 10 > 0$

$(\cot^{-1}x - 5)(\cot^{-1} - 2) > 0$

$\cot^{-1}x \in (-\infty, 2) \cup (5, \infty)$ ...(1)

But $\cot^{-1}x$ lies in $(0, \pi)$

Now, from equation (1)

$\cot^{-1}x \in (0, 2)$

Now, it is clear from the graph

$x \in (\cot 2, \infty)$

17. **(a)** $\text{cat}\left(\sum_{n=1}^{19}\cot^{-1}\left(1 + \sum_{p=1}^{n} 2p\right)\right)$

$$= \cot\left(\sum_{n=1}^{19}\cot^{-1}(1 + n(n+1))\right)$$

$$= \cot\left(\sum_{n=1}^{19}\tan^{-1}\left(\frac{(n+1)-n}{1+(n+1)n}\right)\right)$$

$$\left[\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right) : \text{for } x > 0\right]$$

$$= \cot\left(\sum_{n=1}^{19}(\tan^{-1}(n+1) - \tan^{-1}n)\right)$$

$= \cot(\tan^{-1}20 - \tan^{-1}1)$

$$= \cot\left(\tan^{-1}\left(\frac{20-1}{1+20\times 1}\right)\right)$$

$$= \cot\left(\tan^{-1}\left(\frac{19}{21}\right)\right) = \cot\cot^{-1}\left(\frac{21}{19}\right) = \frac{21}{19}$$

18. **(d)** $x = \sin^{-1}(\sin 10)$

$\Rightarrow x = 3\pi - 10$ $\quad\left\{3\pi - \dfrac{\pi}{2} < 10 < 3\pi + \dfrac{\pi}{2} \Rightarrow 3\pi - x = 10\right.$

and $y = \cos^{-1}(\cos 10)$ $\quad\left\{3\pi < 10 < 4\pi \Rightarrow 4\pi - x = 10\right.$

$\Rightarrow y = 4\pi - 10$

$\therefore y - x = (4\pi - 10) - (3\pi - 10) = \pi$

19. **(a)** $\cos^{-1}\left(\dfrac{2}{3x}\right) + \cos^{-1}\left(\dfrac{3}{4x}\right) = \dfrac{\pi}{2};\left(x > \dfrac{3}{4}\right)$

$$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{3}{4x}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\right) = \sin^{-1}\left(\frac{3}{4x}\right)\left[\because \sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Put $\sin^{-1}\left(\dfrac{3}{4x}\right) = \theta \Rightarrow \sin\theta = \dfrac{3}{4x}$

$$\Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{9}{16x^2}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$$

$$\therefore \cos^{-1}\left(\frac{2}{3x}\right) = \cos^{-1}\left(\frac{\sqrt{16x^2-9}}{4x}\right)$$

$$\Rightarrow \frac{2}{3x} = \frac{\sqrt{16x^2-9}}{4x} \Rightarrow x^2 = \frac{64+81}{9\times 16} \Rightarrow x = \pm\sqrt{\frac{145}{144}}$$

$\Rightarrow x = \frac{\sqrt{145}}{12} \quad \left(\because x > \frac{3}{4}\right)$

20. **(a)** Let $x^2 = \cos 2\theta$; $\Rightarrow \quad \theta = \frac{1}{2}\cos^{-1} x^2$

$\Rightarrow \quad \tan^{-1}\left[\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} - \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]$

$= \left[\frac{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}\right]$

$\Rightarrow \tan^{-1}\left[\frac{1+\tan\theta}{1-\tan\theta}\right] = \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$

$= \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2$

21. **(c)** Given that, $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

$= \tan^{-1} x + 2\tan^{-1} x = 3\tan^{-1} x$

$\tan^{-1} y = \tan^{-1}\left[\frac{3x - x^3}{1-3x^2}\right]$

$\Rightarrow y = \frac{3x - x^3}{1-3x^2}$

22. **(c)** $f(x) = 2\tan^{-1} x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$\Rightarrow f(x) = 2\tan^{-1} x + \pi - 2\tan^{-1} x$

$\Rightarrow f(x) = \pi$

$\Rightarrow f(5) = \pi$

23. **(a)** $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow -\frac{3\pi}{4} \le \left(\sin^{-1} x - \frac{\pi}{4}\right) \le \frac{\pi}{4}$

$0 \le \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 \le \frac{9}{16}\pi^2 \quad ..(1)$

Statement II is true

$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = a\pi^3$

$\Rightarrow (\sin^{-1} x + \cos^{-1} x)\,[(\sin^{-1} x + \cos^{-1} x)^2 - 3\sin^{-1} x \cos^{-1} x] = a\pi^3$

$\Rightarrow \frac{\pi^2}{4} - 3\sin^{-1} x \cos^{-1} x = 2a\pi^2$

$\Rightarrow \sin^{-1} x\left(\frac{\pi}{2} - \sin^{-1} x\right) = \frac{\pi^2}{12}(1-8a)$

$\Rightarrow \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{12}(8a-1) + \frac{\pi^2}{16}$

$\Rightarrow \left(\sin^{-1} x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{48}(32a-1)$

Putting this value in equation (1)

$0 \le \frac{\pi^2}{48}(32a-1) \le \frac{9}{16}\pi^2$

$\Rightarrow 0 \le 32a - 1 \le 27$

$\frac{1}{32} \le a \le \frac{7}{8}$

Statement-I is also true

24. **(a)** Since, x, y, z are in A.P.

$\therefore \quad 2y = x + z$

Also, we have

$2\tan^{-1} y = \tan^{-1} x + \tan^{-1}(z)$

$\Rightarrow \tan^{-1}\left(\frac{2y}{1-y^2}\right) = \tan^{-1}\left(\frac{x+z}{1-xz}\right)$

$\Rightarrow \frac{x+z}{1-y^2} = \frac{x+z}{1-xz} \quad (\because 2y = x+z)$

$\Rightarrow y^2 = xz$ or $x + z = 0 \Rightarrow x = y = z = 0$

25. **(b)** Given $\sin^{-1} x > \cos^{-1} x$ where $x \in (0, 1)$

$\Rightarrow \sin^{-1} x > \frac{\pi}{2} - \sin^{-1} x$

$\Rightarrow 2\sin^{-1} x > \frac{\pi}{2} \Rightarrow \sin^{-1} x > \frac{\pi}{4}$

$\Rightarrow x > \sin\frac{\pi}{4} \Rightarrow x > \frac{1}{\sqrt{2}}$

Maximum value of $\sin^{-1} x$ is $\frac{\pi}{2}$

So, maximum value of x is 1. So, $x \in \left(\frac{1}{\sqrt{2}}, 1\right)$.

26. **(c)** We know that,

$\tan^{-1}\frac{1}{1+2} + \tan^{-1}\frac{1}{1+2\times 3} + \tan^{-1}\frac{1}{1+3\times 4} + ... +$

$\tan^{-1}\frac{1}{1+(n-1)n} + \tan^{-1}\frac{1}{1+n(n+1)} + ... +$

$\tan^{-1}\frac{1}{1+(n+19)(n+20)} = \tan^{-1}\frac{n+19}{n+21}$

$\Rightarrow \tan^{-1}\frac{n-1}{n+1} + \tan^{-1}\frac{1}{1+n(n+1)}$

$+ \tan^{-1}\frac{1}{1+(n+1)(n+2)} + ... + \frac{1}{1+(n+19)(n+20)}$

$= \tan^{-1}\frac{n+19}{n+21}$

$$\Rightarrow \tan^{-1}\frac{1}{1+n(n+1)}+\tan^{-1}\frac{1}{1+(n+1)(n+2)}+......+$$

$$\frac{1}{1+(n+19)(n+20)}=\tan^{-1}\frac{n+19}{n+21}-\tan^{-1}\frac{n-1}{n+1}$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{n^2+n+1}\right)+\tan^{-1}\left(\frac{1}{n^2+3n+3}\right)+......+$$

$$\tan^{-1}\frac{1}{1+(n+19)(n+20)}$$

$$=\tan^{-1}\left(\frac{\frac{n+19}{n+21}-\frac{n-1}{n+1}}{1+\frac{n+19}{n+21}\times\frac{n-1}{n+1}}\right)=\tan^{-1}\frac{20}{n^2+20n+1}=S$$

$$\therefore \tan^{-1}S=\frac{20}{n^2+20n+1}$$

27. **(a)** $\sin(\cot^{-1}(1+x))=\cos(\tan^{-1}x)$

$\Rightarrow \text{cosec}^2(\cot^{-1}(1+x))=\sec^2(\tan^{-1}x)$

$\Rightarrow 1+[\cot(\cot^{-1}(1+x))]^2=1+[\tan(\tan^{-1}x)]^2$

$\Rightarrow (1+x)^2=x^2 \Rightarrow x=-\frac{1}{2}$

28. **(d)** $\sin^{-1}\left(\frac{x}{5}\right)+\text{cosec}^{-1}\left(\frac{5}{4}\right)=\frac{\pi}{2}$

$$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right)=\frac{\pi}{2}-\text{cosec}^{-1}\left(\frac{5}{4}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right)=\frac{\pi}{2}-\sin^{-1}\left(\frac{4}{5}\right)$$

$[\because \sin^{-1}x+\cos^{-1}x=\pi/2]$

$$\Rightarrow \sin^{-1}\left(\frac{x}{5}\right)=\cos^{-1}\left(\frac{4}{5}\right)$$

$$\sin^{-1}\frac{x}{5}=\sin^{-1}\sqrt{1-\left(\frac{4}{5}\right)^2} \quad \left[\because \cos^{-1}x=\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \sin^{-1}\frac{x}{5}=\sin^{-1}\frac{3}{5} \Rightarrow \frac{x}{5}=\frac{3}{5}$$

$\Rightarrow x=3$

29. **(c)** $\cos^{-1}x-\cos^{-1}\frac{y}{2}=\alpha$

$$\Rightarrow \cos^{-1}\left(\frac{xy}{2}+\sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)}\right)=\alpha$$

$$\Rightarrow \cos^{-1}\left(\frac{xy+\sqrt{4-y^2-4x^2+x^2y^2}}{2}\right)=\alpha$$

$$\Rightarrow xy+\sqrt{4-y^2-4x^2+x^2y^2}=2\cos\alpha$$

$$\Rightarrow \sqrt{4-y^2-4x^2+x^2y^2}=2\cos\alpha-xy$$

Squaring both sides, we get

$$\Rightarrow 4-y^2-4x^2+x^2y^2=4\cos^2\alpha+x^2y^2-4xy\cos\alpha$$

$$\Rightarrow 4x^2+y^2-4xy\cos\alpha=4\sin^2\alpha.$$